Energy eigenstates of magnetostatic waves and oscillations

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Effect of excitation of magnetostatic oscillations in a ferrite resonator by the microwave magnetic field was a subject of many publications of more than the last 40 years. The most interesting multiresonance spectrum of absorption peaks one can observe experimentally is a case of disk-form small ferrite resonators. It is shown in this paper that such small ferrite resonators can be considered as ''artificial molecular structures'' with properties characterized by energy eigenstates of magnetostatic oscillations. A special interest in these properties may be found in the field of microwave artificial composite materials.

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I. INTRODUCTION

Some quantum mechanical effects are readily simulated by sufficiently flat microwave resonators since the Schrödinger and the Helmholtz equations are equivalent in two dimensions (see, for example, [1]). The question, however, is if one can use some specific properties of microwave resonators not for simulation, but to obtain fundamentally new physical effects. A special interest in these new effects may be found in the field of artificial composite materials. Recently, we put forward a concept of artificial microwave bianisotropic materials based on a composition of ferrite magnetostatic (MS) resonators with special-form surface metalizations—the magnetostatically controlled bianisotropic materials (MCBMs) [2,3]. The MCBMs, being local temporally dispersive bianisotropic media, demonstrate new electromagnetic properties unknown for any natural materials [3–5].

Fundamental principles of macroscopic electrodynamics of bianisotropic media should arise from the microscopic point of view [5]. It is supposed that the main characteristics of a ferrite bianisotropic particle (BAP) should be based on the effect of the MS oscillations in a ferrite body coupled with the surface electric current oscillations on a region of metallization [2]. Recently, this effect of magnetoelectric coupling in a ferromagnetic resonator with a surface metallization has been experimentally verified [6,7]. Now, in the attempts to solve an excitation problem for characterization of a multiresonance regime in a ferrite BAP, we are faced with the fact that even an excitation problem for MS in a ''pure'' (without a surface metallization) ferrite resonator is rather far from completion.

Effect of excitation of MS oscillations in a ferrite resonator by the rf magnetic field was a subject of many publications of more than the last 40 years. It is well known that the necessary condition for excitation of multiple MS modes in the ferrite ellipsoids (or the ferrite sphere, as a particular case) is that the exciting rf magnetic field at the sample be essentially *nonuniform* [8,9]. In this case, however, one can see just only a few absorption peaks in a spectrum. In his theory of MS oscillations in a ferrite spheroid, Walker had shown, for the first time, that orthogonality relations between different modes of the rf magnetization takes place [10]. These orthogonality relations, however, were not obtained as a result of formulation of the *energy spectral problem* in a ferrite spheroid and, therefore, cannot be used to define the energetic levels of MS oscillations.

The situation can be completely different when nonspherical (nonellipsoidal, more precisely) ferrite samples are used. In a case of small ferrite disks, for example, placed into a region of the *uniform* magnetic field, a long series of oscillating MS modes are excited [11,12]. These experimental results demonstrate that magnetosatic modes actually diagonalize the total magnetic energy. We do not have, however, the proper explanation of so rich a spectrum of the absorption peaks. The role of nonuniform internal dc magnetic field in disk-shaped samples was demonstrated in experiments [13]. Schlömann tried to explain the mechanism of conversion of electromagnetic power into spin-wave power based on the fact that the effective wavelength of the spin waves becomes large in the presence of a suitably nonuniformed dc magnetic field [14]. He found a certain resemblance between equation of motion for the magnetization and the Schrödinger equation. His classical approach, however, cannot give any explanations about an excitation of a rich spectrum, one can observe in experiments with disk-form samples. We do not have an initial formulation of the energy spectral problem in this case.

To explain the effect of coupling between the electromagnetic field and very small ferrite resonators (particles) the mathematical apparatus, similar to the quantum mechanical apparatus and based on the theory of linear operators [15], has to be used. With the use of a simple model of an ''open ferrite disk,'' we will show in this paper that magnetostatic oscillations in a normally magnetized sample can be described by *eigenfunctions with stationary energy eigenstates*. We will have a possibility to formulate an energy spectral problem and to obtain a discrete spectrum of energy levels. So an ''artificial magnetic atom/molecule'' with new properties becomes a subject of investigations. As a subject for future research, the role of the nonuniform dc magnetic field may be analyzed as a potential-energy perturbation of an initial discrete spectrum in a ferrite disk. Excitation of MS oscillations by the rf magnetic field is supposed to be considered as a time-dependent perturbation.

In a case of a ferrite BAP the energetic levels should be defined by coupled oscillations of the rf magnetization and the rf surface electric current. The aim of our future publica-

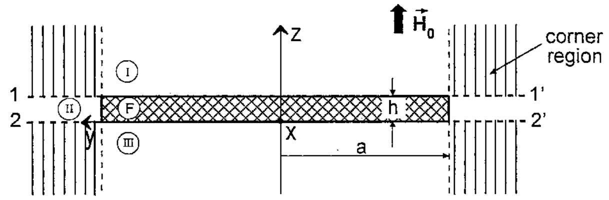

FIG. 1. Ferrite disk with a small thickness/diameter ratio.

tions is an analysis of such a ferrite BAP (that contains two subsystems and can be called conventionally as an ''artificial magnetoelectric molecular structure'') based on the present analysis of eigenfunctions and energy eigenvalues in a ferrite disk. One can expect to have new physical effects in these particles. In particular, recent qualitative investigation demonstrates interesting symmetry properties of a ferrite BAP [16].

II. FERRITE SPHEROID AND FERRITE DISK

Let sizes of a ferromagnetic resonator be much less than the electromagnetic wavelength, but much more than the spin-wave wavelength taking into account the exchange interaction. In this case, the magnetostatic approximation can be successfully used [10]. For the irrotational rf magnetic field of the magnetostatic modes,

$$\vec{H} = -\vec{\nabla}\psi, \tag{1}$$

where ψ is the magnetostatic potential, the rf magnetization $\vec{m}$ is defined as

$$\vec{m} = -\vec{\kappa}(\omega)\cdot\vec{\nabla}\psi, \tag{2}$$

where $\vec{\kappa}$ is a tensor of susceptibility. For a ferromagnetic spheroid with the internal dc magnetic field directed along the z axis, Walker obtained the orthogonality relation for two oscillating MS modes [10]:

$$[\omega^{(\lambda)} + (\omega^{(\nu)})^*]\int_{V_{\text{spheroid}}}[\vec{m}_\perp^{(\lambda)}\times(\vec{m}_\perp^{(\nu)})^*]\cdot\vec{e}_z dv = 0, \tag{3}$$

where $\vec{e}_z$ is a unit vector directed along the z axis and $\vec{m}_\perp$ is the magnetization vector with x and y components. As we have pointed out above, this orthogonality relation was not obtained as a result of formulation of the energy spectral problem in a ferrite spheroid and, therefore, cannot be used to define the energetic levels of MS oscillations. There are other types of orthogonality relations for MS oscillations in a ferromagnetic spheroid that also were not obtained as a result of formulation of the energy spectral problem [17].

Let us consider a normally magnetized ''open ferrite disk'' (Fig. 1). The word ''open'' means that in our model no perfect electric or perfect magnetic walls are used in a general case. On the contrary to a case of a spheroid, when an analytical solution is possible, in the case of an open disk, some additional assumptions have to be made to solve the problem analytically. These assumptions concern the fact of separation of variables and an independent imposition of the boundary conditions on a lateral cylindrical surface and plane surfaces of a disk. In a case of such assumptions, we exclude, in fact, an influence of the edge regions. This does not correspond to the exact electrodynamical conditions, but may give, nevertheless, very satisfactory results. It is clear that the more elongated form of a cross section of a structure, the less an influence of the edge regions we have. In optical waveguides with a rectangular form of a cross section, the method of separation of variables gives the better results of calculations, the less ratio thickness/width (the so-called ''Marcatili approximation'' [18]). One can expect that in our analysis of an ''open ferrite disk,'' a structure with a small axial ratio is more preferable. In connection with our assumption, it is relevant to point out that in the experiments [11,12] we have a multiresonance regime of MS modes just in ferrite disks with a small thickness to diameter ratio (approximately 1/15–1/20).

Based on our model, we will consider a structure shown in Fig. 1 as a section of an open cylindrical MS waveguide with the longitudinal z axis, restricted by two planes $z=0$ and $z=h$. In a case of an axially magnetized cylinder, we have *reciprocal* MS waveguide modes, that is, every mode propagating in the positive direction of the z axis has a counterpart—the same mode propagating in the negative direction of the z axis [19,20]. So, one can consider eigen MS oscillations in a normally magnetized ferrite disk as standing MS waves in a cylindrical waveguide. This fact will allow us to formulate the energy spectral problem for MS oscillations in a disk-form ferrite resonator. For this formulation, however, an initial analysis of the spectral problem for MS waveguide modes has to be done.

III. SPECTRAL PROBLEM FOR MS WAVEGUIDE MODES

Taking into account Eq. (1) for the rf magnetic field, the equation for the rf magnetic flux density

$$\vec{B} = -\vec{\mu}(\omega)\cdot\vec{\nabla}\psi \tag{4}$$

[where $\vec{\mu}(\omega) = \vec{I} + 4\pi\vec{\kappa}(\omega)$ is the tensor of permeability, $\vec{I}$ is the unit matrix] and the equation

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{5}$$

one can write the following operator equation [21]:

$$\hat{L}(\omega)V = 0, \tag{6}$$

where

$$\hat{L}(\omega) \equiv \begin{pmatrix} (\vec{\mu}(\omega))^{-1} & \vec{\nabla} \\ -\vec{\nabla}\cdot & 0 \end{pmatrix} \tag{7}$$

is a differential-matrix operator,

$$V \equiv \begin{pmatrix} \vec{B} \\ \psi \end{pmatrix}, \tag{8}$$

is a vector function included in the domain of definition of the operator $\hat{L}$. Equation (6) describes the field inside a ferrite. Outside of a ferrite medium we have the same equation, but with $\vec{\vec{\mu}}=\vec{\vec{I}}$. Let the fields' variation along with the longitudinal z axis be described by the factor $e^{-\gamma z}$. Based on the Eq. (6) inside a ferrite, analogous equation with $\vec{\vec{\mu}}=\vec{\vec{I}}$ outside a ferrite and taking into account homogeneous boundary conditions, one can formulate a spectral problem for MS waveguide modes with a wave number γ as a spectral parameter. For two MS waveguide modes, we have the orthogonality relation [21]

$$(\gamma_p+\gamma_q^*)\int_S(\hat{R}\tilde{V}_p)(\tilde{V}_q^*)ds=0, \tag{9}$$

where

$$\hat{R}=\begin{pmatrix} 0 & \vec{e}_z \\ -\vec{e}_z & 0 \end{pmatrix}, \tag{10}$$

$\tilde{V}$ is a *membrane* function of the fields in a waveguide defined from the relation

$$V(x,y,z)=\tilde{V}(x,y)e^{-\gamma z}, \tag{11}$$

and S is a square of a waveguide cross section. The norm of mode p is determined as

$$N_p=i\omega\int_S(\hat{R}\tilde{V}_p)(\tilde{V}_p)^*ds=i\omega\int_S(\tilde{\psi}_p\tilde{\vec{B}}_p^*-\tilde{\psi}_p^*\tilde{\vec{B}}_p)\cdot\vec{e}_z ds. \tag{12}$$

This norm (derived by 4) describes the average (on the period $2\pi/\omega$) power flow through a waveguide cross section. Such a statement can be verified by two ways. First, one can be easily persuaded to the fact that a norm written as

$$N_p=\int_S(\tilde{\vec{E}}_p\times\tilde{\vec{H}}_p^*+\tilde{\vec{E}}_p^*\times\tilde{\vec{H}}_p)\cdot\vec{e}_z ds \tag{13}$$

for membrane functions of electric and magnetic fields and corresponding (being divided by 4) to the average (on the period $2\pi/\omega$) power flow of an electromagnetic-wave waveguide mode [22], amounts to the norm (12) for the irrotational magnetic and rotational electric fields [20,23,24]. Second, let us consider Eq. (6) together with the equation complex conjugated with Eq. (6). We can obtain after some transformations:

$$\frac{i\omega}{4}\vec{\nabla}\cdot(\psi\vec{B}^*-\psi^*\vec{B})+\frac{i\omega}{4}[\vec{B}^*\cdot(\vec{\vec{\mu}}(\omega))^{-1}\cdot\vec{B}-\vec{B}\cdot(\vec{\vec{\mu}}^*(\omega))^{-1}\cdot\vec{B}^*]=0. \tag{14}$$

This is an energy balance equation for monochromatic MS waves. The first term in the left-hand side (LHS) of Eq. (14) is the divergence of the power flow density and the second term in the LHS of Eq. (14) is the density of magnetic losses.

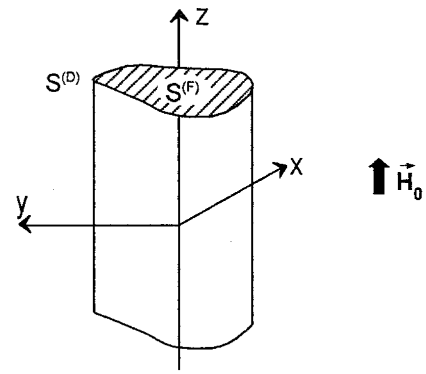

FIG. 2. Axially magnetized ferrite cylinder.

A special feature of a MS waveguide structure based on axially magnetized ferrite cylinder (that does not take place in such types of electromagnetic-wave waveguide structures as closed hollow or open dielectric waveguides) is the fact that in the frequency region $\omega_1\leqslant\omega\leqslant\omega_2$ between two cutoff frequencies $\omega_1=\gamma' H_i$ and $\omega_2=\gamma'[H_i(H_i+4\pi M_s)]^{1/2}$ (where γ' is the gyromagnetic ratio, H_i is the internal dc magnetic field, and M_s is the saturation magnetization), we have a *complete discrete spectrum of propagating* ($\gamma=i\beta$) MS *modes* [19,20].

Together with a system of two first-order homogeneous Eqs. (6). one second-order homogeneous differential equation for a MS waveguide can be considered as well. This is the so-called Walker equation in a ferrite [10]:

$$\hat{G}\psi=0, \tag{15}$$

where

$$\hat{G}=-\vec{\nabla}\cdot(\vec{\vec{\mu}}\vec{\nabla}) \tag{16}$$

is the Walker operator. For a ferrite magnetized along the z axis the tensor of permeability has a form [20]

$$\vec{\vec{\mu}}=\mu_0\begin{bmatrix} \mu & i\mu_a & 0 \\ -i\mu_a & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

where $\mu=1-\omega_1\omega_m/(\omega^2-\omega_1^2)$, $\mu_a=\omega\omega_m/(\omega_1^2-\omega^2)$, $\omega_m=\gamma' 4\pi Ms$.

In this case, Eq. (15) can be rewritten as

$$\mu\frac{\partial^2\psi}{\partial x^2}+\mu\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=0. \tag{18}$$

Based on the Walker equation (15) inside a ferrite and the Laplace equation outside a ferrite and taking into account homogeneous boundary conditions, we consider orthogonality relations for two types of MS waveguide structures: an axially magnetized ferrite cylinder and a normally magnetized ferrite film.

(1) Axially magnetized ferrite cylinder (Fig. 2).

For MS potential written as

$$\psi = \tilde{\psi}(x,y)e^{-\gamma z}. \tag{19}$$

The Walker equation has a form

$$\hat{G}_\perp \psi + \gamma^2 \psi = 0, \tag{20}$$

where

$$\hat{G}_\perp = \mu \nabla_\perp^2. \tag{21}$$

$\nabla_\perp^2$ is the two-dimensional (on the waveguide cross section) Laplace operator.

Let a cross section of a ferrite rod $S^{(F)}$ be surrounded by contour L. For Hermitian tensor $\vec{\mu}$ a double integration by parts gives:

$$\int_{S^{(F)}} (G_\perp \tilde{\psi})\tilde{\psi}^* ds = \int_{S^{(F)}} (G_\perp \tilde{\psi})^* \tilde{\psi}\, ds + \oint_L P^{(F)}(\tilde{\psi},\tilde{\psi}^*) d\ell. \tag{22}$$

The contour integral in Eq. (22) has a form

$$\oint_L P^{(F)}(\tilde{\psi},\tilde{\psi}^*) d\ell = \oint_L (\tilde{B}_n \tilde{\psi}^* - \tilde{B}_n^* \tilde{\psi}) d\ell, \tag{23}$$

where $\tilde{B}_n$ is a component of a membrane function of the magnetic flux density normal to contour L. The validity of expression (23) can be shown with the use of a simple example of a ferrite rod with a rectangular cross section restricted by coordinates x_1, x_2 and y_1, y_2. In this case, one has after a double integration by parts:

$$\oint_L P^{(F)}(\tilde{\psi},\tilde{\psi}^*) d\ell = \left\{ \int_{y_1}^{y_2} \left[\left(\mu \frac{\partial \tilde{\psi}}{\partial x} + i\mu_a \frac{\partial \tilde{\psi}}{\partial y} \right) \tilde{\psi}^* - \left(\mu \frac{\partial \tilde{\psi}}{\partial x} + i\mu_a \frac{\partial \tilde{\psi}}{\partial y} \right)^* \tilde{\psi} \right] dy \right\} \Bigg|_{x_1}^{x_2} + \left\{ \int_{x_1}^{x_2} \left[\left(-i\mu_a \frac{\partial \tilde{\psi}}{\partial x} + \mu \frac{\partial \tilde{\psi}}{\partial y} \right) \tilde{\psi}^* - \left(-i\mu_a \frac{\partial \tilde{\psi}}{\partial x} + \mu \frac{\partial \tilde{\psi}}{\partial y} \right)^* \tilde{\psi}^* \right] dx \right\} \Bigg|_{y_1}^{y_2}. \tag{24}$$

That is, we have an integral in the form of expression (23). Since a line integral can be represented as integrals by coordinate projects, the validity of expression (23) becomes evident for a general case of contour L.

A cross section of a dielectric region $S^{(D)}$ surrounding a ferrite rod is extended to infinity ($x \to \infty$, $y \to \infty$) and is restricted by the inner contour L. We have the Laplace equation in a dielectric region and, in accordance with the Green theorem, one obtains a contour integral

$$\oint_L P^{(D)}(\tilde{\psi},\tilde{\psi}^*) d\ell = \oint_L \left(\frac{\partial \tilde{\psi}}{\partial n} \tilde{\psi}^* - \frac{\partial \tilde{\psi}^*}{\partial n} \tilde{\psi} \right) d\ell, \tag{25}$$

where n is a normal to contour L. Because of homogeneous boundary conditions (the continuity of $\tilde{\psi}$ and $\tilde{B}_n$ on the contour L)

$$\oint_L [P^{(F)}(\tilde{\psi},\tilde{\psi}^*) + P^{(D)}(\tilde{\psi},\tilde{\psi}^*)] d\ell = 0, \tag{26}$$

the following orthogonality relation for two MS waveguide modes takes place:

$$[\gamma_p^2 - (\gamma_q^*)^2] \int_S \tilde{\psi}_p \tilde{\psi}_q^* ds = 0, \tag{27}$$

where $S = S^{(F)} + S^{(D)}$.

(2) Normally magnetized ferrite film (Fig. 3).

For MS potential written as

$$\psi = \tilde{\zeta}(z) e^{-\vec{q}\cdot\vec{r}},$$

where $\vec{r}$ is a radius vector in a plane of a ferrite film, one has the Walker equation

$$\frac{\partial^2 \psi}{\partial z^2} + q^2 \mu \psi = 0. \tag{28}$$

Based on Eq. (28) in a ferrite region and similar equation (with $\mu = 1$) for a dielectric with taking into account homogeneous boundary conditions on surfaces $z=0$ and $z=h$ (the continuity of ψ and $\partial\psi/\partial z$) and the condition: $\psi \to 0$ for $|z| \to \infty$, one obtains the orthogonality relation for two MS modes:

$$[q_a^2 - (q_b^*)^2] \left[\int_0^h \mu \tilde{\zeta}_a \tilde{\zeta}_b^* dz + \int_{-\infty}^0 \tilde{\zeta}_a \tilde{\zeta}_b^* dz + \int_h^\infty \tilde{\zeta}_a \tilde{\zeta}_b^* dz \right] = 0, \tag{29}$$

Two forms of the orthogonality relations (9) and (27) will be used further to obtain the energy eigenstates for modes propagating in a MS waveguide. To analyze the energy

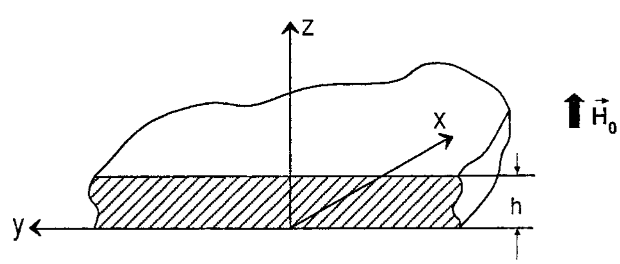

FIG. 3. Normally magnetized ferrite film.

eigenstates in MS ferrite resonator, the orthogonality relation (29) has to be used in addition.

To investigate the energy relations in a MS waveguide, the question about a probability distribution function should be considered.

IV. MS POTENTIAL AS A PROBABILITY DISTRIBUTION FUNCTION

In some cases of MS waveguides, the knowledge of the MS potential wave function ψ gives a possibility to define every state of the physical quantities.

Let us represent a magnetostatic potential in Eq. (19) as

$$\psi_n = A\tilde{\varphi} e^{-\gamma z}, \tag{30}$$

where A is a dimensional coefficient and $\tilde{\varphi}$ is a dimensionless membrane function.

Since the membrane functions of MS modes in an axially magnetized ferrite cylinder give a complete discrete set of functions (on a waveguide cross section), the dimensionless membrane function $\tilde{\varphi}$ in Eq. (30) can be written as

$$\tilde{\varphi} = \sum_{n=1}^{\infty} a_n \tilde{\varphi}_n, \tag{31}$$

where $\tilde{\varphi}_n$ is a membrane function of MS mode. In a case of a cylindrical MS waveguide, $\tilde{\varphi}_n$ are characterized by Bessel functions [19]. Because of the orthogonality relation (27), one can write

$$\int_S |\tilde{\varphi}_n|^2 ds = 1. \tag{32}$$

A system of functions $\tilde{\varphi}_n$ is the orthonormal system of functions and, therefore, we have

$$\int_S |\tilde{\varphi}|^2 ds = \sum_{n=1}^{\infty} |a_n|^2. \tag{33}$$

Amplitude a_n shows with what weight the state $\tilde{\varphi}_n$ is represented in $\tilde{\varphi}$.

Let the function $\tilde{\varphi}$ be normalized to *unity*, that is,

$$\int_S |\tilde{\varphi}|^2 ds = 1, \tag{34}$$

This condition means that

$$\sum_{n=1}^{\infty} |a_n|^2 = 1. \tag{35}$$

If the normalization condition (34) takes place, the function $|\tilde{\varphi}|^2$ is a probability distribution function for a configuration (a waveguide cross section) of a system.

We can introduce a notion of an average quantity (a mean value) $\overline{f}$ of a physical value f [15]:

$$\overline{f} = \sum_n |a_n|^2 f_n, \tag{36}$$

where f_n is an eigenvalue of a quantity f that satisfies the operator equation

$$\hat{f}_\perp \tilde{\varphi}_n = f_n \tilde{\varphi}_n. \tag{37}$$

The operator $\hat{f}_\perp$ acts on the waveguide cross section. We can see that

$$\overline{f} = \int_S \tilde{\varphi}^* (\hat{f}_\perp \tilde{\varphi}) ds. \tag{38}$$

Since operator $\hat{f}$ is self-conjugated, one can easily obtain

$$(f_n - f_m) \int_S \tilde{\varphi}_n \tilde{\varphi}_m^* ds = 0. \tag{39}$$

In particular, f_m and f_n may be eigenvalues of the normalized MS energy.

V. ENERGY EIGENSTATES OF MS MODES IN AN AXIALLY MAGNETIZED FERRITE CYLINDER

Let us represent the MS potential as a quasimonochromatic quantity,

$$\psi = \psi^{(\max)}(t) e^{i\omega t}, \tag{40}$$

where the amplitude $\psi^{(\max)}$ is a smooth function of a time, so that

$$\left| \left(\omega^{-1} \frac{\partial}{\partial t} \right) \psi^{(\max)} \right| \ll \psi^{(\max)}, \tag{41}$$

Let a part of an infinitely long lossless MS waveguide be restricted by two cross sections placed at $z = z_1, z_2$. For the quasimonochromatic MS wave process, one can write the energy balance equation in a waveguide section:

$$\int_{z_1}^{z_2} \int_S \vec{\nabla}_\parallel \cdot \overline{\vec{P}_\parallel} ds\, dz + \frac{d}{dt} \int_{z_1}^{z_2} \int_S \overline{w} \quad ds\, dz = 0, \tag{42}$$

where $\overline{\vec{P}_\parallel}$ is the average (on the rf period) power for flow density along a MS waveguide, $\nabla_\parallel$ means the longitudinal part of divergence, and $\overline{w}$ is the average (on the rf period) density of the energy. Based on Eqs. (4), (14), and (17), the first term in the LHS of Eq. (42) is written as

$$\begin{aligned} &\int_{z_1}^{z_2} \int_S \vec{\nabla}_\parallel \cdot \overline{\vec{P}_\parallel} ds\, dz \\ &= \tfrac{1}{4} i\omega\mu_0 \int_{z_1}^{z_2} \int_S \vec{\nabla}_\parallel \cdot (\psi \vec{\nabla}_\parallel \psi^* - \psi^* \vec{\nabla}_\parallel \psi) ds\, dz \\ &= \tfrac{1}{4} i\omega\mu_0 \int_{z_1}^{z_2} \int_S (\psi \dot{\nabla}_\parallel^2 \psi^* - \psi^* \dot{\nabla}_\parallel^2 \psi) ds\, dz, \end{aligned} \tag{43}$$

where $\vec{\nabla}_{\parallel}$ and $\nabla_{\parallel}^2$ are one-dimensional (longitudinal) parts of, respectively, the gradient and the Laplace operators.

By appropriate change of variables, any system of equations describing oscillations in one-dimensional linear systems with distributed parameters may be written as [25]:

$$\hat{Q}\vec{u}=\frac{\partial \vec{u}}{\partial t}, \tag{44}$$

where $\vec{u}(z,t)$ is a vector function with components u_1, u_2, ... describing system properties and $\hat{Q}=\hat{Q}(z)$ is a differential matrix operator. In our case of a MS waveguide sections, oscillations are described by distribution of MS potential ψ with respect to the longitudinal z axis. This distribution is characterized by the second-order one-dimensional differential equation. When we rewrite Eq. (44) as

$$-\frac{i}{X}\nabla_{\parallel}^2\psi=\frac{\partial \psi}{\partial t}, \tag{45}$$

(where X is a constant quantity) we can see that based on Eqs. (42) and (45), one obtains

$$\begin{aligned}\tfrac{1}{4}i\omega\mu_0\int_{z_1}^{z_2}\int_S(\psi\nabla_{\parallel}^2\psi^*-\psi^*\nabla_{\parallel}^2\psi)ds\,dz\\ =\frac{\omega\mu_0 X}{4}\int_{z_1}^{z_2}\int_S\left(\psi\frac{\partial\psi^*}{\partial t}+\psi^*\frac{\partial\psi}{\partial t}\right)ds\,dz\\ =\frac{d}{dt}\int_{z_1}^{z_2}\int_S\overline{w}ds\,dz.\end{aligned} \tag{46}$$

The average energy of a MS waveguide section can be characterized as

$$\overline{W}=\frac{w\mu_0 X}{4}\int_{z_1}^{z_2}\int_S\psi\psi^*ds\,dz. \tag{47}$$

One can see that coefficient X has the dimension $[\mathrm{T\,L^{-2}}]$ or (in SI units) sec/M^2. This coefficient may be defined by the following way.

Since

$$\nabla_{\parallel}^2\psi-\gamma^2\psi=0,$$

we have from Eq. (45) for MS waveguide mode n:

$$-iX_n\frac{\partial\psi_n}{\partial t}+\gamma_n^2\psi_n==0. \tag{48}$$

In a case of a pure monochromatic wave process characterized by frequency $\omega(e^{i\omega t})$, one has for mode n

$$X_n=-\frac{\gamma_n^2}{\omega}. \tag{49}$$

We define a notion of the *normalized average (on the* rf *period)* MS *energy of propagating waveguide mode* $n(\gamma_n=i\beta_n)$, E_n, as the average (on the rf period) MS energy in a waveguide section with the unit length and the unit characteristic cross section. Based on expression (30), (31), (47), and (49), one obtains for mode n with the unit amplitude ($|a_n|^2=1$)

$$E_n=g\frac{\mu_0}{4}\beta_n^2, \tag{50}$$

where g is the unit dimensional coefficient with the same dimension as coefficient A^2.

Our definition of an average quantity (mean value) [see expression (38)] allows to write

$$\overline{E}=\int_S\tilde{\varphi}^*(\hat{F}_{\perp}\hat{\varphi})ds,$$

where $\hat{F}_{\perp}$ is the *operator* of the normalized average (on the rf period) MS energy of a propagating waveguide mode. The following operator equation takes place:

$$\hat{F}_{\perp}\tilde{\varphi}_n=E_n\tilde{\varphi}_n. \tag{51}$$

Based on Eq. (18) and taking into account expression (31), we have

$$\mu\nabla_{\perp}^2\tilde{\varphi}_n=\beta_n^2\tilde{\varphi}_n.$$

Since E_n is proportional to β_n^2, the operator $\hat{F}_{\perp}$ has to be proportional to $\mu\nabla_{\perp}^2$. So we can write

$$\hat{F}_{\perp}=K\mu\nabla_{\perp}^2, \tag{52}$$

where K is a constant value. We have for a propagating mode

$$K=g\frac{\mu_0}{4}. \tag{53}$$

For MS mode n, we have the differential equation

$$g\frac{\mu_0\mu}{4}\nabla_{\perp}^2\tilde{\varphi}_n=E_n\tilde{\varphi}_n. \tag{54}$$

Similarly to expression (27), one can easily obtain from Eq. (54):

$$(E_n-E_{n'})\int_S\tilde{\varphi}_n\tilde{\varphi}_{n'}^*ds=0. \tag{55}$$

This property of orthonormality of MS waveguide modes is one of the most important characteristics of eigenfunctions in the energy spectrum.

The solution for ψ_n may be written as:

$$\psi_n=Aa_n\tilde{\varphi}_n e^{-i2[(|E_n|/g\mu_0)z]^{1/2}}, \tag{56}$$

similarly to the solution obtained from the one-dimensional time-independent Schrödinger equation for the wave function of a free particle [15]. Operator $\hat{F}_{\perp}$ acting to the MS potential resemble the Hamiltonian operator acting to the wave function [15].

A special feature of a MS waveguide structure based on an axially magnetized ferrite cylinder, as it was pointed out above, is the fact that in the frequency region between two cutoff frequencies, ω_1 and ω_2, we have a complete discrete spectrum of propagating MS modes [19,20]. In accordance with Eq. (50), every propagating mode of a spectrum is characterized by the normalized average (on the rf period) MS energy. So for given frequency ω, the total normalized average (on the rf period) MS energy for all the spectrum of propagating modes can be written as:

$$E_{\text{total}}=\frac{\mu_0}{4}g\sum_{n=1}^{\infty} a_n^2\beta_n^2. \tag{57}$$

VI. ENERGY EIGENSTATES OF MS OSCILLATIONS IN A FERRITE DISK

The energy spectral problem solved for a MS waveguide is very important for energy spectral problem in MS resonators. Let us consider an ‘‘open ferrite disk’’ shown in Fig. 1. We have four main regions: region F—a ferrite and regions I–III—dielectrics. The role of the corner regions is supposed to be neglected. We describe the MS potential in a ferrite by the Walker equation (15). Outside a ferrite, we have the Laplace equation. The boundary conditions at surfaces of a disk are the continuity of MS potential ψ and the normal components of magnetic flux density $\vec{B}$. Further, $\psi\to 0$ at infinity. For the dc magnetic field directed along the z axis, the Walker equation in cylindrical coordinates (ρ,α,z) has a form:

$$\mu\left(\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\alpha^2}\right)+\frac{\partial^2\psi}{\partial z^2}=0. \tag{58}$$

Outside a ferrite, we have the similar equation, but with $\mu=1$.

In cylindrical coordinates acceptable solutions for the Walker and Laplace equations in different regions of the structure, are of the form:

$$\psi=\psi(\rho)\psi(\alpha)\psi(z).$$

In every region of the structure we have the following solution for $\psi(\alpha)$:

$$\psi(\alpha)=C_\alpha e^{-im\alpha}, \tag{59}$$

where m is an integer (positive or negative). For different regions of coordinate z, the function $\psi(z)$ has the following forms:

(1) for regions F and II $(0\leqslant z\leqslant h)$:

$$\psi^{(F)}(z)=C_z^{(F)}\cos(\beta^{(F)}z)+D_z^{(F)}\sin(\beta^{(F)}z). \tag{60}$$

(2) for region I $(z\geqslant h)$:

$$\psi^{(\mathrm{I})}(z)=C_z^{(\mathrm{I})}e^{-\beta^{(D)}(z-h)}. \tag{61}$$

(3) for region III $(z\leqslant 0)$:

$$\psi_{(z)}^{(\mathrm{III})}=C_z^{(\mathrm{III})}e^{\beta^{(D)}z}. \tag{62}$$

In expressions (60)–(62), $\beta^{(F)}$ and $\beta^{(D)}$ are, respectively, the wave numbers along the z axis in a ferrite and a dielectric $[\gamma^{(F)}=i\beta^{(F)},\gamma^{(D)}=\beta^{(D)}]$. It is necessary to point out that the solution in regions F and II $(0\leqslant z\leqslant h)$ has the form of expression (60) only for reciprocal (with respect to the z axis) MSWs. In a case of an axially magnetized cylinder, the MSWs are reciprocal waves [19].

To find out the solutions for $\psi(\rho)$ we have to substitute into Eq. (58) the solution (59) for $\psi(\alpha)$ and the solutions (60)–(62) for $\psi(z)$. For region I $(z\geqslant h,\rho\leqslant a)$ we have

$$\frac{\partial^2\psi(\rho)}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi(\rho)}{\partial\rho}+\left[(\beta^{(D)})^2-\frac{m^2}{\rho^2}\right]\psi(\rho)=0. \tag{63}$$

The same equation, one has for region III $(z\leqslant 0,\rho\leqslant a)$. For region $F(0\leqslant z\leqslant h,\rho\leqslant a)$ we have

$$\frac{\partial^2\psi(\rho)}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi(\rho)}{\partial\rho}-\left[\frac{(\beta^{(F)})^2}{\mu}+\frac{m^2}{\rho^2}\right]\psi(\rho)=0. \tag{64}$$

In region II $(0\leqslant z\leqslant h,\rho\geqslant a)$, we can write

$$\frac{\partial^2\psi(\rho)}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi(\rho)}{\partial\rho}-\left[(\beta^{(F)})^2+\frac{m^2}{\rho^2}\right]\psi(\rho)=0. \tag{65}$$

One can see that Eq. (64) becomes the Laplace equation if $(\beta^{(F)})^2/\mu$ is replaced by $(\beta^{(F)})^2$. Equations (64) and (65) correspond to equations obtained and analyzed by Joseph and Schlömann for MSWs in a long, axially magnetized cylinder [19].

In the Bessel equation (63), $\beta^{(D)}$ is real and for the problem under investigation, only $J_m(\beta^{(D)}\rho)$—the Bessel function of real argument is a physically acceptable solution. So for region I we have

$$\psi^{(\mathrm{I})}(\rho)=C_\rho^{(\mathrm{I})}J_m(\beta^{(D)}\rho). \tag{66}$$

A similar expression, we have for region III:

$$\psi^{(\mathrm{III})}(\rho)=C_\rho^{(\mathrm{III})}J_m(\beta^{(D)}\rho). \tag{67}$$

A physically acceptable solution for Eq. (64) is possible only for $\mu<0$. In this case, we have

$$\psi^{(F)}(\rho)=C_\rho^{(F)}J_m(\beta^{(F)}(-\mu)^{1/2}\rho). \tag{68}$$

For Eq. (65) one obtains

$$\psi^{(\mathrm{II})}(\rho)=C_\rho^{(\mathrm{II})}K_m(\beta^{(F)}\rho), \tag{69}$$

where K_m is the Bessel function of an imaginary argument.

Now let us impose the boundary conditions on the planes $z=0$ and $z=h$. We have

$$\psi^{(\mathrm{I})}(z)=\psi^{(F)}(z)\big|_{z=h},\quad \frac{\partial\psi_{(z)}^{(\mathrm{I})}}{\partial z}=\frac{\partial\psi_{(z)}^{(F)}}{\partial z}\bigg|_{z=h} \tag{70}$$

and

$$\psi^{(F)}(z)=\psi^{(\mathrm{III})}(z)|_{z=0}, \quad \frac{\partial \psi^{(F)}_{(z)}}{\partial z}=\frac{\partial \psi^{(\mathrm{III})}_{(z)}}{\partial z}\bigg|_{z=0}. \tag{71}$$

The boundary conditions (70), (71) together with solutions (60)–(62) give a system of homogeneous equations for the coefficients. The condition of equality of a determinant of this system of equations to zero leads to the following transcendental equation:

$$\tan(\beta^{(F)}h)=-\frac{2\sqrt{-\mu}}{1+\mu}. \tag{72}$$

Equation (72) was obtained based on the relation

$$\beta^{(D)}=\frac{1}{\sqrt{-\mu}}\beta^{(F)}, \tag{73}$$

derived from Eq. (58).

At the lateral surface ($\rho=a, 0\leqslant z\leqslant h$) one should have the continuity conditions for potential ψ and for the radial component of magnetic flux density $\vec{B}$. Taking into account expression (17) for tensor $\overleftrightarrow{\mu}$ and making necessary vector transitions from rectangular to cylindrical coordinates, one obtains for the interior of a ferrite

$$B_\rho=\mu_0(\mu H_\rho+i\mu_\alpha H_\alpha)=\mu_0\left(-\mu\frac{\partial\psi}{\partial\rho}-i\mu_\alpha\frac{1}{\rho}\frac{\partial\psi}{\partial\alpha}\right), \tag{74}$$

where H_ρ and H_α are the radial and circumferential components of the rf magnetic field.

The continuity conditions for B_ρ and ψ on the boundary $\rho=a$ then leads to the following equation

$$(-\mu)^{1/2}\frac{J'_m}{J_m}+\frac{K'_m}{K_m}-\frac{\mu_a m}{|\beta^{(F)}|a}=0, \tag{75}$$

where we denoted

$$J_m\equiv J_m(|\beta^{(F)}|(-\mu)^{-1/2}a); \quad K_m\equiv K_m(|\beta^{(F)}|a),$$

$$J'_m\equiv\frac{\partial J_m(|\beta^{(F)}|(-\mu)^{-1/2}\rho)}{\partial\rho}\bigg|_{\rho=a};$$

$$K'_m\equiv\frac{\partial K_m(|\beta^{(F)}|\rho)}{\partial\rho}\bigg|_{\rho=a}. \tag{76}$$

To obtain eigenfrequencies of a ferrite disk resonator one has to solve a system of two Eqs. (72) and (75) for given values of *h*, *a*, and *m*. The solutions in forms of relations (60) and (66)–(68) are correct only for $\mu<0$. It means that the admissible frequency region is restricted as $\omega_1\leqslant\omega\leqslant\omega_2$.

In our model of an ''open ferrite disk,'' the MS potential distribution with respect to the *z* axis can be characterized by equations similar to Eq. (45), but with different coefficients *X* in ferrite and delectric regions. By analogy with expression (46), one obtains

$$\frac{\omega\mu_0}{4}\int_S\left[X^{(D)}\int_{-\infty}^0\left(\psi^*\frac{\partial\psi}{\partial t}+\psi^*\frac{\partial\psi}{\partial t}\right)dz+X^{(F)}\int_0^h\left(\psi^*\frac{\partial\psi}{\partial t}+\psi^*\frac{\partial\psi}{\partial t}\right)dz+X^{(D)}\int_h^\infty\left(\psi\frac{\partial\psi^*}{\partial t}+\psi^*\frac{\partial\psi}{\partial t}\right)dz\right]ds$$
$$=\frac{d}{dt}\int_S\left[\int_{-\infty}^0\overline{w_1^{(D)}}dz+\int_0^h\overline{w^{(F)}}dz+\int_\infty^0\overline{w_2^{(D)}}dz\right]ds. \tag{77}$$

The average (on the rf period) MS energy of a ferrite resonator can be characterized as

$$\overline{W}=\frac{\omega\mu_0}{4}\int_S\left[X^{(D)}\int_{-\infty}^0\psi\psi^*dz+X^{(F)}\int_0^h\psi\psi^*dz+X^{(D)}\int_h^\infty\psi\psi^*dz\right]ds. \tag{78}$$

We represent a resonance mode in a ferrite disk as follows:

$$\psi_{pq}=A\tilde{\xi}_p(z)\tilde{\varphi}_q(\rho,\alpha), \tag{79}$$

where $\tilde{\xi}$ and $\tilde{\varphi}$ are piecewise continuous dimensionless functions that are defined based on Eqs. (60)–(62) and (66)–(69). For function $\tilde{\varphi}_q$ the normalization condition (32) takes place. At resonance frequency ω_{pq} [found based on Eqs. (72) and (75)], one can write the normalization condition for function $\tilde{\xi}_p$ taking into account expression (29):

$$\int_0^h\mu_{pq}|\tilde{\xi}_p|^2dz+\int_{-\infty}^0|\tilde{\xi}_p|^2dz+\int_h^\infty|\tilde{\xi}_p|dz=1, \tag{80}$$

where we denoted $\mu_{pq}\equiv\mu(\omega_{pq})$.

For resonance frequency ω_{pq}, one has for coefficients $X^{(D)}$ and $X^{(F)}$ in expression (78):

$$X^{(D)}_{pq}=\frac{(\beta^{(D)}_{pq})^2}{\omega_{pq}} \tag{81}$$

and [see expression (73)]

$$X^{(F)}_{pq}=\frac{(\beta^{(F)}_{pq})^2}{\omega_{pq}}=-\mu_{pq}\frac{(\beta^{(D)}_{pq})^2}{\omega_{pq}}. \tag{82}$$

Here we denoted $\beta_{pq}\equiv\beta(\omega_{pq})$.

Based on the normalization conditions (32) and (80), we introduce the notion of the *normalized* average (on the rf period) energy of MS oscillations. For a ferrite disk resonator with a unit characteristic volume, one has the normalized energy of the MS oscillation with the unit amplitude:

$$E_{pq}=g\frac{\mu_0(\beta^{(D)}_{pq})^2}{4}. \tag{83}$$

Here expression (78), (81), and (82) have been used; g is the unit dimensional coefficient.

Based on expression (83) and taking into account Eq. (18) in a ferrite region and the Laplace equation outside a ferrite, one can obtain

$$g\frac{\mu_0}{4}\nabla_\perp^2\tilde{\varphi}_q=E_{pq}\tilde{\varphi}_q. \tag{84}$$

It means that the operator equation

$$\hat{F}_\perp\tilde{\varphi}_q=E_{pq}\tilde{\varphi}_q \tag{85}$$

with

$$\hat{F}_\perp=g\frac{\mu_0}{4}\nabla_\perp^2 \tag{86}$$

takes place.

The energy orthogonality condition for MS oscillations in a ferrite disk resonator obtained from Eq. (84) has a form:

$$(E_{pq}-E_{pq'})\int_S\tilde{\varphi}_q\tilde{\varphi}_{q'}^*ds=0. \tag{87}$$

Our analysis of energy eigenstates of MS oscillations in a ferrite disk is valid only when the wave-number (with respect to z axis) spectrum of ''thickness modes'' $\tilde{\xi}_p(z)$ [see expression (79)] is ''rare'' enough compared to the ''dense'' spectrum of ''in-plane modes'' $\tilde{\varphi}_q(\rho,\alpha)$. This situation really takes place in our case of a ferrite disk with a small thickness to diameter ratio ($h/2a\ll 1$). So, for the main ''thickness mode'' one has an *energetic spectrum* due to the ''in-plane mode'' spectrum.

It is also necessary to call the reader's attention to a very important fact that, in accordance with Joseph and Schlömann analysis [19], one has different absolute values of wave numbers for the left-hand and right-hand circularly polarized MS waves in a ferrite rod. This becomes clear from Eq. (75) that is dependent on a sign of integer m. In our analysis, this fact leads to differences of energies for the left-hand and right-hand circularly polarized MS oscillations in a ferrite disk.

VII. DISCUSSION

We have shown in this paper that in a MS waveguide structure based on an axially magnetized ferrite cylinder, propagating modes are characterized by quantities of the normalized average (on the rf period) MS energy with the orthogonality property of eigenfunctions in the energy spectrum. With the use of a simple model of an ''open ferrite disk,'' we have shown in this paper that magnetostatic oscillations in a normally magnetized sample can be described by eigenfunctions with stationary energy eigenstates. These results of our theoretical analysis show that propagating MS modes and MS oscillations actually can diagonalize the magnetic energy.

In classical waveguide problems (for electromagnetic-wave [22] and, in particular, MS-wave waveguides [20,24]), an excitation of normal modes by the external (given) currents and charges was analyzed. In particular, the traditional technique describes an excitation of MS-wave waveguides due to electric current transducers [20,24]. In [21], we considered another type of excitation: the MS mode excitation due to the external (given) rf magnetic field. The excitation problem analyzed in [21] is not, however, so well justified. One has only the homogeneous (Walker) equation for MS potential, but there are no such kind of a nonhomogeneous equation with the external rf magnetic field in the right-hand side. Now, based on the results of this paper, one has a possibility to consider the MS mode excitation as a time-dependent perturbation of the energy spectrum in an axially magnetized ferrite cylinder.

Our analysis of energy eigenstates of MS oscillations gives a possibility to explain a multiresonance spectrum of absorption peaks, one can experimentally observe in the effect of coupling between the rf magnetic field and very small ferrite-disk resonators (particles). It becomes clear now that the observed multiresonance peaks are due to portion absorption of energy of the exciting rf magnetic field. In our analysis, we used the mathematical apparatus based on the theory of linear operators similar to the quantum mechanical apparatus.

With respect to the problem under consideration, it is important to keep in mind the fact that when in classical electrodynamics structures the spectral problems are characterized by *wavenumbers* and *frequencies* as the spectral parameters, in quantum mechanics structures there are *energy eigenstates* as the spectral parameters.

With use of the spectral method (with energy eigenstates) we are able now to develop the perturbation theory for MS oscillations. These should be time-independent perturbations (to take into account the role of nonuniform internal dc magnetic field) and time-dependent perturbations (to consider excitation by the rf magnetic field). Similar to the quantum mechanical problems [15], the perturbation method for MS oscillations constitutes a separate treatment and should be a subject for future efforts.

Ferrite disk, contrary to a ferrite sphere, has cylindrical symmetry. Such a type of symmetry characterizes the dipole field similarly to the field of a two-atomic molecule. So, MS oscillations in a small ferrite-disk resonator can be similar to dynamical processes that take place in a two-atomic molecule. An artificial magnetic medium composed by small MS ferrite-disk resonators can bear a resemblance to a paramagnetic material. It is interesting to note that a bianisotropic particle based on a ferrite-disk resonator with a special-form surface metallization, being considered as a combination of two (electric and magnetic) dipoles, has symmetry properties similar to the properties one can observe in a case of elementary particles: a combination of change conjugation (C), parity (P), and time reversal (T) (the so-called CPT-invariance) [16].

One can see that one-dimensional wave equation (45) contains a first derivative with respect to time, and a second derivative with respect to the space coordinate. So there is an asymmetry between the time and space coordinates and, therefore, Eq. (45) is not invariant with respect to the Lorentz transformations. The reason of this asymmetry is that the magnetostatic wave equations in a ferrite, being resulted in the ''distorted'' Maxwell equations written for irrotational magnetic and rotational electric fields, describe ''slow'' wave processes with velocities much smaller than the velocity of electromagnetic waves at the same frequency [20,24]. Similarly, the Schrödinger wave equation is nonrelativistic: it is suitable only for particles whose velocity is much smaller than the velocity of light [15].

VIII. CONCLUSION

At present, we are witnesses to a very strong interest in electromagnetic complex (anisotropic, chiral, bianisotropic) materials. Artificial composite materials play an important role in attempts to realize new electromagnetic materials. It becomes clear, however, that to have such materials with properties that satisfy the principles of macroscopic electrodynamics, two levels of consideration, microscopic and macroscopic, have to be used [5,26–28]. Microscopic properties of natural electromagnetic materials are based on quantum mechanical theory. In this paper we have shown that some particles in *artificial* composite materials can be considered microscopically as ''*artificial molecular structures*'' with properties characterized by energy eigenstates of oscillations. Similar to the theory of natural condense media [26,27], further development of the theory of artificial dense materials should be focused on the energetic-spectrum properties of a system of artificial molecules.

New definite physical results concerning the subject of this paper arise from recent experimental study of spectra in ferrite resonators with special-form surface electrodes [29,30]. An adequate description of the observed regular multiresonance spectra of magnetoelectric oscillations excited by the external rf electric, magnetic, and combined (electric+magnetic) fields, should, certainly, be given based on the *quantized picture* with a proper consideration of the magnetostatic-potential functions as the probability functions.

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